Attractors in Chaos

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Abstract

Most of the systems occurring in nature are chaotic i.e. changing initial conditions can lead to vast changes in future. In this document we study two chaotic systems: The Logistic Map and The Lorenz Attractor. The first shows how chaos can appear even in simple systems. We summarize and visualize how changing parameters causes this system to show chaotic behavior. Next we plot the Lorenz System and observe chaotic behavior beyond certain parameter values. Rather than a rigorous explanation of this phenomenon we opted a more visual and programming heavy treatment.

1 Some background

For want of a nail, the shoe was lost;
For want of a shoe, the horse was lost;
For want of a horse, the rider was lost;
For want of a rider, the battle was lost;
For want of a battle, the kingdom was lost!

Anonymous

Dynamics is the study of time evolutionary processes and the corresponding system of equations is known as a **dynamical system**. Such evolutionary processes may have another property called determinacy: if the entire future and past can be determined from the present state uniquely its said to be a deterministic system, if this determination is not unique its called semi-deterministic system and if no such determination exists then its called non-deterministic system. The evolutionary process may also be either a continuous time process (given by differential equations) or a discrete time process (given by difference equations). If the evolution is governed by a linear differential equation(s) (continuous time) or difference equation(s) (discrete time) the dynamical system is called a linear dynamical system. If the evolution is governed by a non-linear differential equation(s) or difference equation(s) we call the system a non-linear dynamical system.

A **phase space** for a dynamical system is hypothetical space in which all possible states of a system evolve. For example consider the dynamical system

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n); i \in \{1, 2, \ldots, n\} \tag{1}$$

Here x_i's depend on parameter t, then we consider an abstract space with coordinates $(x_1, \cdots, x_n)$. The solution points $(x_1(t), \cdots, x_n(t))$ of the differential equation(s) correspond to a path in this space as t varies. These solution paths are called **trajectories** or **orbits** and this space is called phase space. A **fixed point** (or **equilibrium point**) for the above system is $(x_1^*, \ldots, x_n^*)$ such that $f_i(x_1^*, \ldots, x_n^*) =$

0 for all $i \in \{1, \dots, n\}$. Fixed points give a simple and important form of orbit. We say that the fixed/equilibrium point is a **source** when nearby solutions tend away from it. The equilibrium point is a **sink** when nearby solutions tend toward it. Sources are often called unstable fixed points and sinks stable.

Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions. By aperiodic long-term behavior, we mean that the orbits don't end up either periodic or circling around some equilibrium point. An **attractor** is an invariant set to which all nearby orbits converge. They are the sets that one "sees" when a dynamical system is iterated on a computer. A **strange attractor** is an attractor that exhibits sensitive dependence on initial conditions. We shall see below two dynamical systems that suffer chaos, they will help to "see" these definitions better.

2 The Logistic Map

> As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.
>
> Albert Einstein

A simple model for population growth of organisms is

$$\frac{dP}{dt} = rP$$

where $P(t)$ is population at time t and $r > 0$ is growth parameter. Solving it one sees that the model predicts exponential growth ($P(t) = P_0 e^{rt}$, P_0 is population at $t = 0$). Clearly this model implies overpopulation and what not. Population biologists tried tackling this by assuming that $\frac{P'}{P}$ decreases when P becomes large enough, but this creates an additional problem that when population is above certain capacity (say C is the population at which they exhaust all resources and are unable to develop further) the growth rate becomes negative. Pierre Verhulst in 1838 suggested that $\frac{P'}{P}$ should decrease linearly with P. Thus the logistic model of population

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{C}\right)$$

was born. It describes the self-limiting growth of a biological population pretty well.

In 1976 Robert May showed that even simple non-linear maps can show chaotic behavior using logistic maps, a discrete analog of the logistic equation above.

$$x_{n+1} = rx_n(1 - x_n) \tag{2}$$

where $x_n \geq 0$ is a dimensionless measure of population in n^{th} generation and $r \geq 0$ is the intrinsic growth rate. We will restrict r in $[0, 4]$ as then (2) maps the interval $[0, 1]$ onto itself. For other values its not very interesting, as it can be shown that if $x_n > 1$ for some n then future iterations will diverge to $-\infty$ which means that the population goes extinct (not a very useful thing to study we guess). [Suppose $x_n = 1 + \delta$ for some $\delta > 0$ then $x_{n+1} - x_n = -(r + \delta)(1 + \delta) < -r$ thus sequence decreases by at least r every turn.]

The logistic map becomes a chaotic system for certain values of the parameter r. We shall state (without proofs) some results about convergence of the population with logistic growth for various ranges of parameter.

- For $r < 1$, we get $x_n \to 0$ as $n \to \infty$.
- For $1 < r < 3$, the population grows and eventually reaches a non-zero steady state.
- At somewhere above $r = 3$, a period 2-cycle is born i.e. a large population in one generation is followed by a smaller one in next and this repeats. For example when $r = 3.2$ the long-term behavior of the orbit is to get pulled to an attracting cycle of period 2. Increasing r more will give period 2^n cycles in the x_n vs n plot (for that parameter). If r_n denotes the parameter value at which the first occurrence of a period-2^n cycle happens, then using numerical computations we see that $\lim_{n\to\infty} r_n = r_\infty = 3.57\dots$

- For $r_\infty < r < 4$, in words of Robert May, "*What the Christ happens for $r > r_\infty$?*". Once we plot the bifurcation diagram we will have a great picture of the situation. We will be able to see the chaotic behavior above r_∞. This marks the onset of chaos.

2.1 Bifurcation

In context of dynamical systems, bifurcation theory studies changes in orbit, number of solutions to the system, etc., when the parameter(s) are changed. A bifurcation diagram provides a way to see these bifurcations and summarizes in one single picture all the possible stable, long-term behaviors of the system. It shows the values visited or approached asymptotically as the bifurcation parameter(s) are changed.

In order to plot the bifurcation diagram for the logistic map, we follow these steps: Start with $r = 0$ and set initial population value $x_0 = \frac{1}{2}$. Iterate the map 10000 times and plot the last 20 values, and discard the rest. Then increase r by a small amount and repeat. The attractor (for Logistic Map) for any value of r can be seen by drawing a line at that r.

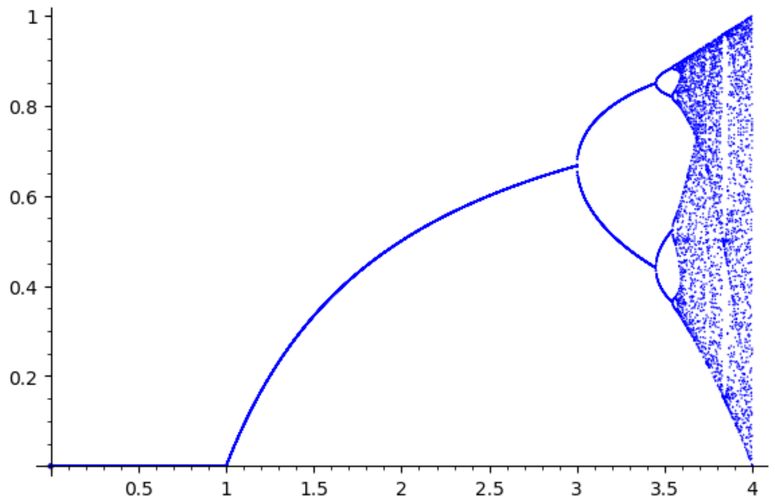

Figure 1: Bifurcation diagram of the Logistic Map [Refer to Code File - 1]

As can be seen in Figure 1, the region $3 < r < 4$ is such an intricate one. As we already noted the periods start doubling as r increases beyond 3.

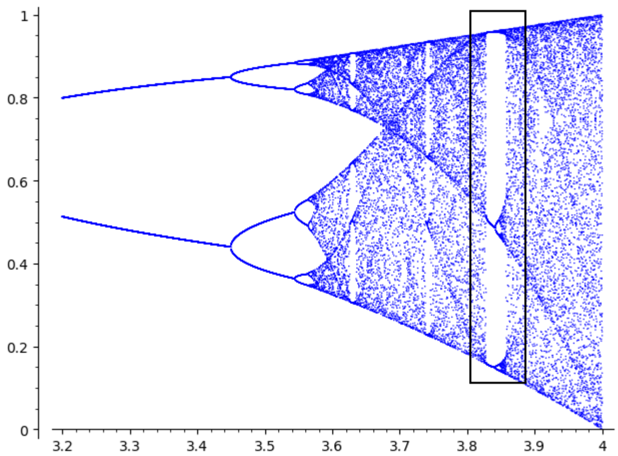

Figure 2: Emergence of order from Chaos [Refer to Code File - 2]

We expect them to go increasing but something strange happens. At around 3.57 the branching becomes chaotic (after continuous 'doubling') and we see almost solid vertical lines. As we go beyond this r_∞, one can see periodic "windows" among these dotted vertical lines. For example at around 3.83 a window of period 3 opens up [See Fig. 2 [1]] (the blank space that appears around 3.8, note that there are three branches over there which go on coupling). But after that again a 'doubling' in period starts and we enter yet another region of chaos after sometime. It can be shown that there are infinitely many such periodic openings. There is indeed a pattern to the appearance of these windows which we are unable to discuss (look up Sharkovsky Ordering) that's why we see the pitchfork motif being repeated. The bifurcation diagram of the Logistic map is indeed a diagram of remarkable complexity, one can see more and more structure upon zooming in.

Logistic map is indeed a chaotic system as even slight changes in initial conditions, say if we plot x_n vs n plots ($r = 3.9$) with one starting at $x_0 = 0.5$ and another at $x_0 = 0.5000001$ the orbits start nearby but soon diverge by a lot as can be seen in the picture below.

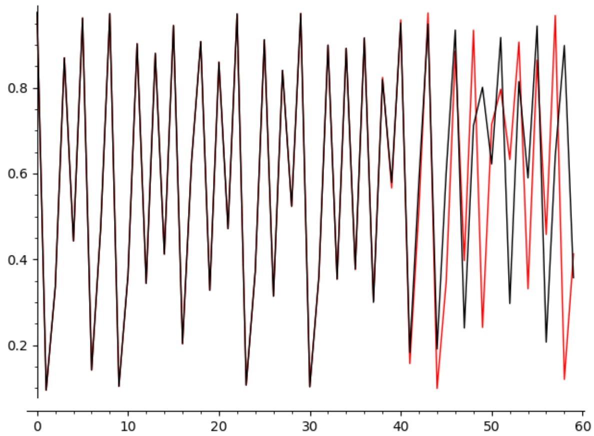

Figure 3: Sensitivity to initial conditions in the Logistic Map [Refer to Code File - 3]

2.2 Logistic Map and SAGE

Explanation of code for Fig. 1 To plot the bifurcation diagram we start by defining three functions:

`chaos(r,x)` - In this function, `x` is iterated according to the law $x_n = rx_{n-1}(1 - x_{n-1})$. We send the last 20 values of `x` to the array `l` after 10000 iterations.

`plot(r,l)` - For a given value of r this plots the values in `l`. We use `list_plot()` to plot the points from `l`. We initialize the plot `p` by plotting the points `(0,l[0])`. Then we add to this plot all the points from the array `l` using `p+=list_plot([(r,l[i])])`.

`chaos2(x)` - The function takes `x` as input, initial population and for each value of `r` between 0 and 4 (both included) in step values of $\frac{1}{1000}$. For each `r` in the loop we call the functions `chaos(r,x)` and `plot(r,l)` to plot the values. All the plots throughout the range of `r` are combined and returned.

Explanation of code for Fig. 3 In this plot we try to see the difference when we start with two slightly different population values. We implement the previously defined `chaos(r,x)` function for `r=3.9`. We execute the plot by using the in-built SAGE function: `p=list_plot(l,plotjoined=True,color='red')` to get a continuous plot.

[1] This was plotted by the authors themselves and not copied from the web; in-fact all images are produced by the authors!

3 Lorenz System

> If the single flap of a butterfly's wings can be instrumental in generating a tornado, so also can all the previous and subsequent flaps of its wings, as can the flaps of the wings of millions of other butterflies, not to mention the activities of innumerable other more powerful creatures including our own species
>
> Edward Lorenz

In 1963 Edward Lorenz, a professor at MIT attempted to set up a system of differential equations that would explain (to some extent) the unpredictable nature of weather. He used a very simplified model of fluid convention which of course was far from reality but ended up giving unexpected results. He looked at a two-dimensional fluid cell which was heated from below and cooled from above. The equations for this system are themselves very complicated so he made vastly simplifying assumptions about it. He was led to the following three-dimensional system of differential equations in three parameters.

$$\frac{dx}{dt} = \sigma(y - x) \tag{3}$$
$$\frac{dy}{dt} = x(\rho - z) - y \tag{4}$$
$$\frac{dz}{dt} = xy - \beta z \tag{5}$$

The parameters σ, ρ, β are positive real parameters. σ and ρ are proportional to the Prandtl number and Rayleigh number respectively. The system is non-linear (presence of xz and xy), non-periodic and deterministic. Also if (x, y, z) is a solution then so is $(-x, -y, z)$.

This system has three fixed points the origin, $Q_+ = (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1)$ and $Q_- = (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1)$ (can be seen by simple algebra). When $\rho < 1$ it can be shown that the system has only O (origin) as a fixed point. And when $1 < \rho < \rho^* := \sigma\left(\frac{\sigma+\beta+3}{\sigma-\beta-1}\right)$ then both $Q_\pm$ are sinks. Beyond ρ^* a special kind of bifurcation happens (called Hopf Bifurcation) where these stable fixed points lose their stability. But the trajectories don't diverge to infinity with time! The orbits are bounded and non-periodic.

Lorenz system is indeed chaotic as we will see now. Lorenz considered the parameters $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$ (clearly $\rho > \rho^*$). We start with initial points $(x, y, z) = (0, 1, 1)$. The trajectory starts and swings directly into one of the fixed points $Q_\pm$ and then starts spiraling outwards and goes to the other fixed point, again takes some rounds and goes back to the previous one and repeats. The number of circuits it takes around either of these points is in fact random! This gives the trajectory in the phase space the look of **butterfly wings**. The orbits indeed stay in this shape (for any initial condition) hence its an attractor, but the motion along the attractor is chaotic. This is called the Lorenz attractor, and is a strange attractor (see end of this section). This system is sensitive to initial conditions, say if we plot with initial conditions $(x, y, z) = (0.2, 1, 1)$ although in beginning they will give close enough orbits, after some time they will be significantly different. So it's not really possible to make accurate predictions about the future of Lorenz equations. One can be certain that the orbit will remain on the attractor, even though one is very uncertain what exact path the orbit will follow as it weaves its way across the attractor.

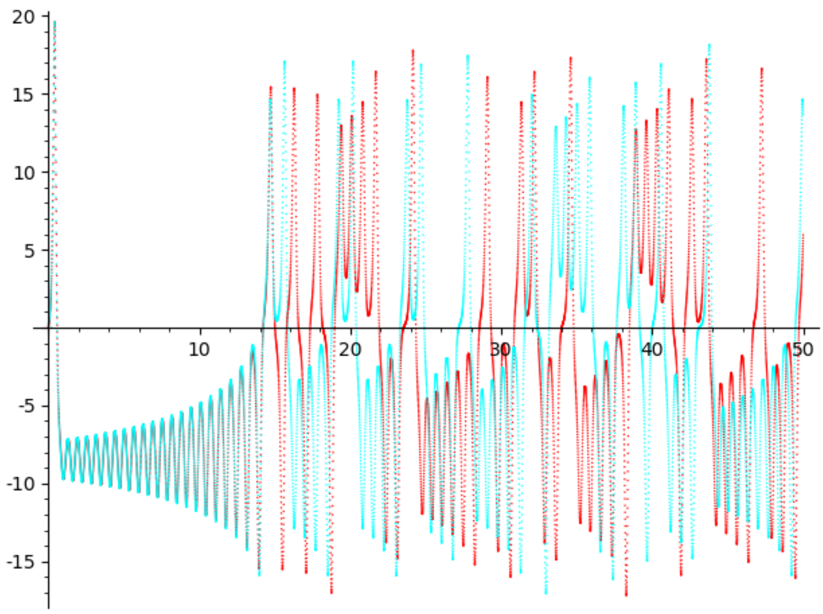

Figure 4: Sensitivity to initial conditions in Lorenz Attractor: The different colors mark different initial values of $x(0)$ [Refer to Code File - 4]

Below we have plotted the attractor for $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$.

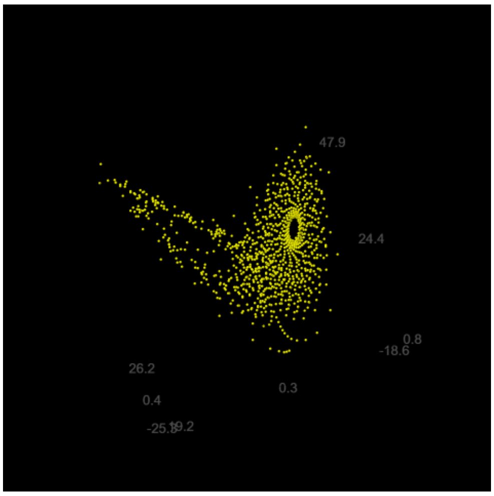

Figure 5: Lorenz Attractor with fewer time points [Refer to Code File - 5]

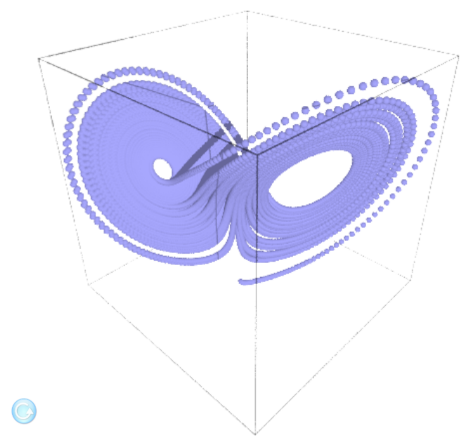

Figure 6: Lorenz Attractor with more time points; plotted with time difference 0.005 [Refer to Code File - 5]

We mentioned that the Lorenz equations are an over-simplified version of weather models. But they are not entirely useless from practical point of view. H. Haken derived the Lorenz equations in 1975 while studying the problem of irregular spiking in lasers, Edgar Knobloch in 1981 derived it from disc dynamos, etc.

We mentioned that Lorenz attractor is a strange attractor, it's no easy thing to show. It was Stepehen Smale's 14th Problem on his list of unsolved problems. Mathematicians lacked a rigorous proof that exact solutions of the Lorenz equations will resemble the shape generated above, and they also could not prove that its a strange attractor (or that even if it's chaotic!). It was one of the milestones in history of dynamical systems when Warwick Tucker in 2002 showed that Lorenz system is indeed a strange attractor. His paper was titled "The Lorenz attractor exists".

3.1 Lorenz Attractor and SAGE

Explanation of code for Lorenz Attractor For solving the Lorenz system of differential equations we use the in-built SAGE command:
`sage.calculus.desolvers.desolve_odeint(des, ics, times, dvars, rtol=None, atol=None)`
which solves numerically a system of first-order ordinary differential equations using `odeint` from `scipy.integrate` module.

We take a look at the parameters we have used:

- `des` - right hand sides of the system is given by `lorenz=[sigma*(y-x),x*(rho-z)-y,x*y-beta*z]`, a list.
- `ics` - initial conditions which we are taking as $0, 1, 1$ for x, y, z respectively.
- `times` - a sequence of time points in which the solution must be found. We are working in time range between 0 and 50(both included) with difference between any two time points being 0.005.
- `dvars` - the dependent variables in the same order as `des,[x,y,z]`.
- `rtol, atol` - The input parameters `rtol` and `atol` determine the error control performed by the solver. The solver will control the vector e, of estimated local errors in y, according to an inequality of the form: max-norm of $\frac{\mathtt{e}}{\mathtt{ewt}} \leq 1$, where `ewt` is a vector of positive error weights computed as: $\mathtt{ewt} = \mathtt{rtol} \times \mathtt{abs(y)} + \mathtt{atol}$. `rtol` and `atol` can be either vectors of the same length as y or scalars. We have set `rtol` and `atol` as e^{-13} and e^{-14} respectively.

Explanation of code for Fig. 4 The output is a `numpy.ndarray` with the solution of the system at each time in `times`, which is stored in `sol`. To access the values of x, y, z from the array `sol` we define `x=sol[:,0], y=sol[:,1], z=sol[:,2]` where `sol[:,i]` gives us the i^{th} column of the array. The parameters of the differential equation σ, ρ, β are taken as inputs using the sliders.

Plotting the system: We will use `list_plot` to plot the points from `x,y,z`. We initialize the plot `p` by plotting the points x[0], y[0], z[0]. Then we add to this plot all the points from the array `sol` using `p+=list_plot([(x[i],y[i],z[i])],color='black')`.

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