Motivic Homotopy

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This note goes over the basic notions of schemes and gives a minimal working knowledge of motivic homotopy. I have just sourced stuff here mainly for my reference.

1 Basic algebraic geometry

In algebraic geometry $\text{Spec}(X) = (\text{Spec}(X), \mathcal{O}_{\text{Spec}(X)})$ where $\text{Spec}(X)$ us a topological space and $\mathcal{O}$ is a sheaf of rings. We have the isomorphism of hom sets $\text{hom}(\text{Spec}(R), X) \cong \text{hom}(\mathcal{O}_X, R)$. The functor of points view:

$$\begin{aligned}\text{Spec}(X) : \text{Crings} &\to \text{Set}\\ R &\mapsto \text{hom}(X, R)\end{aligned}$$

$$\begin{aligned}\text{Rings}^{op} &\xrightarrow{\simeq} \text{Category of schemes}\\ R &\mapsto \text{Spec}(R)\\ \mathcal{O}_X &\leftarrow\!\shortmid X\end{aligned}$$

"Affine spaces are formal duals of rings"

If X is any functor from Ring to Sets, then $\mathcal{O}_X =$ set of all natural maps $X \to \mathbb{A}^1$. This gets a ring structure. Recall $\mathbb{A}^1 = \text{Spec}(\mathbb{Z}[t])$

$$\begin{aligned}\text{hom}(-, \text{Spec}(R)) : (Aff)^{op} &\to Sets\\ \text{Spec}(S) \simeq Y &\mapsto [Y, \text{Spec}(R)] = [Spec(S), Spec(R)] \cong [R, S]\end{aligned}$$

Each ring morphism $B \to A$ defines a morphism of schemes $\text{spec}A \to \text{spec}B$ and conversely all morphisms between them arise in this fashion. We take as defintion of Spec

$$\begin{aligned}\text{Spec}(R) : Cring &\to Sets\\ S &\mapsto [R, S]\end{aligned}$$

Example 1. The two common schemes that we encounter are $\mathbb{A}^1(R) = \text{Sm}_k(\text{Spec}(R), \text{Spec}(k[t])) = k-alg(k[t], R) = R$ and $\mathbb{G}_m := \text{spec}\, k[t, t^{-1}] = \mathbb{A}^1_k \setminus \{0\}$

2 Motivic Spaces

We take Sm_S and embed them into the presheaf category as it is well-behaved. Zariski topology has too few covers. étale topology has enogh geometry but infinite cohomological dimension. So we use Nisnevich which is finer than Zariski but coarser than étale.

Theorem 2. A presheaf $F : \mathrm{Sm}_S^{op} \to S$ is a Nisnevich shea iff

(1) $F(\Phi) \simeq *$

(2) Let

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d, "p"] \\ U \arrow[r, "i"'] & X \end{tikzcd}$$

be a Nisnevich square in Sm_S, i.e. it is carteisan i is an open immersion, p is étale and $p^{-1}(X-U) \to X-U$ is an isomorphism.Then the square

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(W) \end{tikzcd}$$

is a pullback square.

Also the Yoneda embedding $\mathrm{Sm}_S \to \mathrm{Pre}(\mathrm{Sm})_S$, sends Nisnevich squares to pushout squares.

Definition 3. A presheaf $F \in \mathrm{Pre}(\mathrm{Sm})_S$ is called $\mathbb{A}^1$-invariant if for all $X \in \mathrm{Sm}_S$ the map $\mathrm{pr}^* : F(X) \to F(X \times \mathbb{A}^1)$ is an equivalence.

Definition 4. $\mathrm{Spc}S \subset \mathrm{sPre}(\mathrm{Sm})_S$ is the full sub category of $\mathbb{A}^1$-invariant Nisnevich sheaves. This is the ∞-category of motivic spaces. So we have

$$\mathrm{MotSpc} = L_{\mathbb{A}^1}\mathrm{Sh}_{Nis}(\mathrm{Sm}_S)$$

Here is a remark that I haven't made sense of yet: $\mathrm{Spc}(S)$ is not an ∞-topos.

Definition 5. A map $H : X \times \mathbb{A}^1 \to Y$ in Sm_S is called $\mathbb{A}^1$-homotopy.

We write $\mathbb{G}_m$ for the pointed S-scheme $(\mathbb{A}^1 - \{0\}, 1)$.

The squares

$$\begin{tikzcd} \mathbb{G}_m \times \mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}_m \times \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \times \mathbb{G}_m \arrow[r] & \mathbb{A}^2 - \{0\} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r, "\not\ni"'] & \mathbb{P}^1 \end{tikzcd}$$

are Nisnevich squares. So we obtain pushout squares

$$\begin{tikzcd} \mathbb{G}_m \times \mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \mathbb{G}_m \arrow[d] \\ \mathbb{G}_m \arrow[r] & \mathbb{A}^2 - \{0\} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & * \arrow[d] \\ * \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

in $\mathbf{Spc}(S)$ after contracting $\mathbb{A}^1$. We deduce $\mathbb{A}^2 - \{0\} \simeq \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$ and $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ in $\mathbf{Spc}(S)_*$.

Definition 6. For integers $s \geq w \geq 0$, the motivic spheres are $S^{s,w} = S^{s-w} \wedge \mathbb{G}_m^w \in \mathrm{Spc(S)}_*$. So $\mathbb{P}^1 \simeq S^{2,1}$ and $\mathbb{A}^2 - \{0\} \simeq S^{3,2}$.

There is an equvialence $S^{2n-1,n} \simeq \mathbb{A}^n - \{0\}$.

Definition 7. The category $(\mathcal{SH}(S), \otimes)$ is the initial presentably symmetric monoidal category under $\mathrm{Spc}(S)_*$, i.e. it comes with a functor $\Sigma^\infty : \mathrm{Spc}(S)_* \to \mathcal{SH}(S)$, on which tensoring with $S^{s,w} = \Sigma^\infty S^{s,q}$ defines an equivalence $\Sigma^{s,w} = S^{s,w} \otimes - : \mathcal{SH}(S) \to \mathcal{SH}(S)$. We invert $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$

Theorem 8. Let

$$\mathrm{Spc}(S)_*[(\mathbb{P}^1)^{-1}] = \mathrm{colim}\left(\mathrm{Spc}(S)_* \xrightarrow{(-)\wedge\mathbb{P}^1} \mathrm{Spc}(S)_* \xrightarrow{(-)\wedge\mathbb{P}^1} \mathrm{Spc}(S)_* \to \cdots\right)$$

in Pr^L.

3 Realization

If X is a scheme over k and R is in $k-\mathrm{CAlg}$. Then $X(R)$ is the set of R valued points of X is the set of k-morphisms $\mathrm{Spec}(R) \to X$. If k is a field and $R = k$, the set of k morphisms identifies with the set of points $x \in X$.

A k-valued point on X is $\underline{a} = (a_1, \dots, a_n) \in k^n$ such that $f_i(\underline{a}) = 0, \forall i$. Thus

$$\hom_{kAlg}(A,k) \cong \hom_{Sm_k}(\mathrm{Spec}(k), X)$$

So if X is an S-scheme and T also an S-scheme, then the set of T-points on X is $X(T) := \hom_S(T,X)$.

More generally, if X and Y are schemes, a morphism $Y \to X$ is a Y valued point of X. If $Y = \mathrm{Spec}A$ is affine then the set of Y valued points of X is just $\hom(Y,X)$ and the functor $Y \mapsto \hom(Y,X)$ is called the functor points of the scheme X, since to a given scheme Y it assigns the set of Y-valued points

Example 9. $\mathbb{A}^1(R) = \mathrm{Sm}_k(\mathrm{Spec}(R), \mathrm{Spec}(k[t])) = k - alg(k[t], R) = R$. This also tells us that $\mathbb{A}^n_k(R) = R^n$.

Example 10. The scheme $\mathbb{G}_m := \mathrm{spec}\, k[t,t^{-1}] = \mathbb{A}^1_k \setminus \{0\}$ and $\mathbb{G}_m(R) = R^\times$.

Let $X \in \mathsf{Sm}_k$. The *real points of* X form a space, denoted by $X(\mathbb{R})$, with underlying set $\mathsf{Sm}_k(\mathrm{Spec}(\mathbb{R}), X) \simeq \mathsf{Sm}_{\mathbb{R}}(\mathrm{Spec}(\mathbb{R}), X \times_k \mathrm{Spec}(\mathbb{R}))$, and topology induced as follows: since X is of finite type over k, we may pick an open cover of X by affine subsets of the form $U = \mathrm{Spec}(k[X_1, \dots, X_n]/(f_1, \dots, f_r)) \to \mathrm{Spec}(k)$. Then, the topology on

$$\mathsf{Sm}_k(\mathrm{Spec}(\mathbb{R}), U) \simeq \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid f_i(x_1, \dots, x_n) = 0\ \forall i \leq r\}$$

is induced by the Euclidean one. This construction extends to a functor $(-)(\mathbb{R}) : \mathrm{Pre}(\mathrm{Sm})_S \to \mathrm{Spc}$.

Example 11. $\mathbb{R}$ maybe viewed as the C_2-fixed points of the conjugation action on $\mathbb{C}$. Adopting an equivariant perspective allows one to compute the real realization from the complex one; or more precisely as the geometric C_2-fixed points of some genuine equivariant spectrum. The complex realization is $r_{\mathbb{C}}(T) \simeq \mathbb{S}^1 \wedge \mathbb{C} - \{0\} \simeq \mathbb{S}^2$.

4 Presheaves with transfer

Let X, Y be algebraic varieties over F a field and assume that X is smooth. An elementary multi-valued map is an irreducible subvariety $W \subset X \times Y$ such that $W \to X$ is finite and surjective. A multi-valued map is a $\mathbb{Z}$-linear combination of multi-valued maps. The group of multi-valued maps is denoted by $\mathrm{Cor}(X,Y)$. Cor_F is the category where objects are smooth algebraic varieties and morphisms are multi-valued maps. A presheaf with transfer is a contravariant additive functor $\mathcal{F} : \mathrm{Cor}_F \to Ab$. $\mathbb{Z}_{\mathrm{tr}}(X)(U) = \mathrm{Cor}(U,X)$. We have $\mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}_F) = \mathbb{Z}$.

With this in mind we'll define the group $\mathrm{Cor}(X,Y)$, which will be our morphisms between X,Y in the category Cor_k, to be the what we would get if we allowed multivalued functions between schemes. That is, if a function defines a graph which is an isomorphism with the first factor under the projection, a correspondence will define a subscheme of the product which is finite and surjective under the projection to the first factor. There is a faithful embedding $\mathrm{Sm}_k \hookrightarrow \mathrm{Cor}_k$

Motivic cohomology admits more functoriality than just the contravariant functoriality of schemes, which we encode by $\mathrm{Cor}(S)$. This is reminiscent of the wrong-way maps in classical algebraic topology, called *transfers*. For any map of spaces $X \to Y$ with finite and discrete homotopy fibers we get a map, Becker-Gottlieb transfer,

$\Sigma^\infty_+ Y \to \Sigma^\infty_+ X$, which sends a point $y \in Y$ to the sum of points of the fiber. We import these to the motivic setting using the notion of correspondence of spaces.

Any map $f : X \to Y$ in Sm_S admits a graph Γ_f, defining a functor $\Gamma : \mathrm{Sm}_S \to \mathrm{Cor}(S)$. Cor is additive and symmetric monoidal. Γ induces by Yoneda extension a functor $\mathrm{sPre}(\mathrm{Sm})_S \to \mathrm{Pre}(\mathrm{Cor})(S)$ which is an algebra through Day convolution. With this we can define $\mathrm{Spc}_{\mathrm{tr}}(S) = \mathrm{Pre}^{\Sigma}(\mathrm{Cor}(S)) \otimes_{\mathrm{sPre}(\mathrm{Sm}_S)} \mathrm{Spc}_*(S)$.

Let $\mathrm{Spc}_{\mathrm{tr}}(S) \subset \mathrm{Pre}(\mathrm{Cor})_S$ with Nisnevich sheaves and $\mathbb{A}^1$-invariance be defined as the category of sheaves with transfers on Cor_S. There is a forgetful functor

$$U : \mathrm{Pre}^{\mathrm{tr}}(\mathrm{Cor})_{\mathrm{S}} \to \mathrm{sPre}(\mathrm{Sm}_{\mathrm{S}})$$

This has a left adjoint called the transfer functor denoted by $\mathbb{Z}^{\mathrm{tr}}$ or R^{tr} [1]. We define the stable homotopy category

$$SH_{tr}(S,R) = \mathrm{Spc}_{\mathrm{tr}}(S)[(\mathbb{Z}^{tr}\mathbb{P}^1)^{-1}] \simeq \mathrm{Spc}_{\mathrm{tr}}(S) \otimes_{\mathrm{Spc}_*(S)} SH(S)$$

Putting this all together, the motivic picture looks like this:

$$\begin{tikzcd}
\mathrm{Sm}_S \arrow[r] \arrow[dd] & \mathrm{Pre}(\mathrm{Sm})_S \arrow[r] \arrow[dd, shift left, "\mathbb{Z}^{tr}"] & Spc(S) \arrow[r, shift left, "\Sigma^\infty_{\mathbb{P}^1}"] \arrow[dd, shift left, "\mathbb{Z}^{tr}"] & SH(S) \arrow[l, shift left, "\Omega^\infty_{\mathbb{P}^1}"] \arrow[dr, shift left, "H\mathbb{Z}\wedge -"] \arrow[dd, shift left, "\mathbb{Z}^{tr}"] & \\
 & & & & \mathrm{Mod}_{H\mathbb{Z}} \arrow[ul, shift left, "H\mathbb{Z}"] \arrow[dl, shift left] \\
\mathrm{Cor}_S \arrow[r] & \mathrm{Pre}(\mathrm{Cor})_S \arrow[r] \arrow[uu, shift left, "U"] & Spc_{tr}(S) \arrow[r, shift left, "\Sigma^\infty_{\mathbb{P}^1}"] \arrow[uu, shift left, "U"] & SH_{tr}(S) \arrow[l, shift left, "\Omega^\infty_{\mathbb{P}^1}"] \arrow[ur, shift left] \arrow[uu, shift left, "U"] &
\end{tikzcd}$$

5 Eilenberg-Maclane Spectrum, Motivic Cohomology

Let $HR = U_{\mathrm{tr}}(\mathbb{1}_{\mathcal{SH}_{\mathrm{tr}}(S,R)})$. We have an adjunction

$$\mathrm{Mod}_{HR} \underset{\Psi}{\rightleftarrows} \mathrm{Mod}_{\mathbb{1}}(\mathcal{SH}_{\mathrm{tr}}(S,R)) = \mathcal{SH}_{\mathrm{tr}}(S,R)$$

The naive EM spectrum represents cohomology but doesn't work for Dold-Thom!

Let A be an R-module and $X \in \mathrm{Sm}_S$ be essentially smooth over a field k. Then,

$$H^{p,q}(X,A) \cong [\Sigma^\infty_+ X, \Sigma^{p,q} HA]$$

The fancier version using transfers

$$H^{p,q}(X;\mathbb{Z}) = H^{p-q}_{Zar}(Z, \mathbb{Z}_{tr}(\mathbb{G}_m^{\otimes q})[-q])$$

References

Notes by Qi Zhu

Presheaves with transfer

Motivic Cohomology

https://mathoverflow.net/questions/259894/why-presheaves-with-transfer

Motivic cohomology

On the real realization of the motivic spectrum *ko*

[1] It should all be $\mathrm{Cor}(S,R)$ for correspondences of schemes over S with values taken in R-modules. If we let it take values in Ab the $R = \mathbb{Z}$. But I dropped R everywhere.