Serre Spectral Sequences

Narendran E

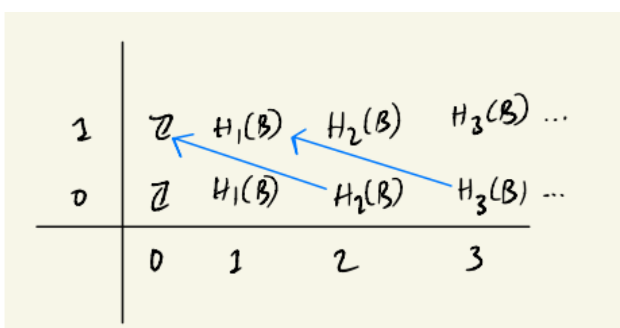
The best way to learn spectral sequences is through calculating a lot of examples. Here I elaborate on a few examples from Hatcher's note

Theorem. *Let $F \to X \to B$ be a fibration with B path connected . If $\pi_1(B)$ acts trivially on $H_*(F;G)$, then there is a spectral sequence*

$$E^2_{p,q} \approx H_p(B; H_q(F;G)) \implies H_n(X;G)$$

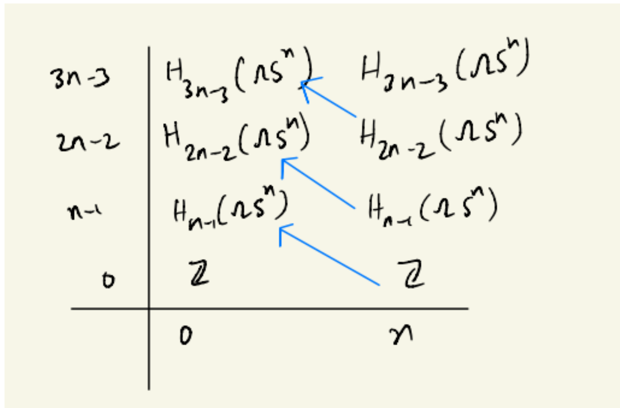
The convergence is the usual strong convergence. So the stable terms $E^\infty_{p,n-p}$ are isomorphic to the successive quotients F^p_n/F^{p-1}_n in a filtration $\{F^i_n\}$ of $H_n(X;G)$ (This notion of convergence is called as strong convergence).

Example 1. We calculate the homology of $K(\mathbb{Z},2)$. Note that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$ space. Consider the pathspace fibration $F \to P \to B$ where B is a $K(\mathbb{Z},2)$and P is the space of paths in B starting at the basepoint. So P is contractible and $F = K(\mathbb{Z},1)$. B is simply connected so we can apply Serre specseq on the fibration. So P is contractible and $F = K(\mathbb{Z},1)$. B is simply connected so we can apply Serre specseq on the fibraton. $h_i(F;\mathbb{Z}) = \mathbb{Z}$ for $i = 0,1$ and 0 otherwise. This gives us the E^2 page:



$d_3, d_4, \cdots = 0$ as they go upward at least 2 rows. So we have $E^3 = E^4 = \cdots = \infty$ except $\mathbb{Z}$ at $(0,0)$ position which survives till E^∞. This gives that d_2 must be isomorhism except for $d_2 : \mathbb{Z}(at(0,0)) \to 0$. This is because any element in the kernel or cokernel of one of these differntials would give a non-zero entry in the E^3 page. Inductively we can finish our argument. $H_1(B) = 0$ from isomorphiscm from $(1,0)$ to the 0 at $(-1,1)$. Similarly $H_2(B) \approx \mathbb{Z}$. and $H_i(B) \approx H_{i-2}(B)$ for $i > 2$.

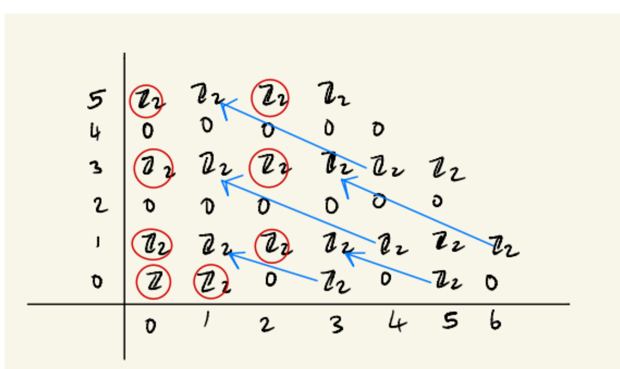
Example 2. We now calculate the homology of ΩS^n by looking at the fibration $\Omega S^n \to P \to S^n$. By application of Whitehead's theorem , ΩS^1 has contractible components. So assume $n \geq 2$, so S^n is simply connected. We have it's E^2 page as in the figure.



As before the E^∞ page has $\mathbb{Z}$ at $(0,0)$. Thee only non0-zero differntial is d_n. This implies we have $E^2 = E^3 = \cdots = E^n$ and $E^{n+1} = \cdots = E^\infty$. So the d_n must be isomorphisms, except the map $d_n : \mathbb{Z}(at(0,0)) \to 0$. It follows that $H_i(\Omega S^n;\mathbb{Z}) = \mathbb{Z}$ for $i \cong 0 \mod n-1$ and 0 otherwise.

Remember to hydrate 💧.

Example 3. $1 \to A \to B \to C \to 1$ be a shortexseq of groups. Look at the induced map $K(B,1) \to K(C,1)$. Converting this map to a fibration, we get $K(A,1) \to K(B,1) \to K(C,1)$ a fiber sequence. Consider the fibration asssociated to the sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$. $\mathbb{R}P^\infty$ is a $K(\mathbb{Z},2)$ space and hence $H^n(\mathbb{R}P^\infty;\mathbb{Z})$ is $\mathbb{Z}_2$ or 0 for $n > 0$ and $H_0 = \mathbb{Z}$.So π_1 acts trivially. ($\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_2)$ gives only the trivial action.)
If the fibration has $K(\mathbb{Z}_2,1) \times K(\mathbb{Z}_2,1)$ as total space, from Kunneth formula we can get that all differentials are zero in E^2 page.
So we look at the fibration with total space $K(\mathbb{Z}_4.1)$. The terms along the diagonal $p+q=n$ are the successive quotients for some filtration of $H_n(K(\mathbb{Z}_4,1);\mathbb{Z})$ which is $\mathbb{Z}_4$ if n is odd and 0 if n is even. So all the $\mathbb{Z}_2$ in the even diagonals must become 0 and in odd diagonals all $\mathbb{Z}_2$ but two must become 0 - $E^\infty_{0,n}$ and $E^\infty_{2,n-2}$.



The $n = 1$ diagonal has no nontrivial differential, so both $\mathbb{Z}_2$ survives till E^∞. The $\mathbb{Z}_2$ in the $n = 2$ diagonal must disapper and this can happen only if it is hit by diff from $\mathbb{Z}_2$ at $(3, 0)$. This leaves two $\mathbb{Z}_2$'s in $n = 3$ diagonal which must survive to E^∞, so there can be no nonzero diff from $n = 4$ diagonal. This pattern continues. So we have the only terms in the $E\infty$ page are the cirled ones in the picture of E^2 page.

The Serre specseq satisfies naturality properties. Suppose we are given two fibrations and a map between them:

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ F' & \longrightarrow & X' & \longrightarrow & B' \end{array}$$

Suppose the two fibrations satisfy the same hypothesis as before, we have the following naturality properties:

1. There are induced maps $f_*^r : E_{p,q}^r \to E_{p,q}'^r$ commuting with differentials, with F_*^{r+1} the map on homlogy induced by f_*^r.

2. The map $\tilde{f}_* : H_*(X; G) \to H_*(X'; G)$ preserves filtrations, inducing a map on successive quotient groups which is the map f_*^∞.

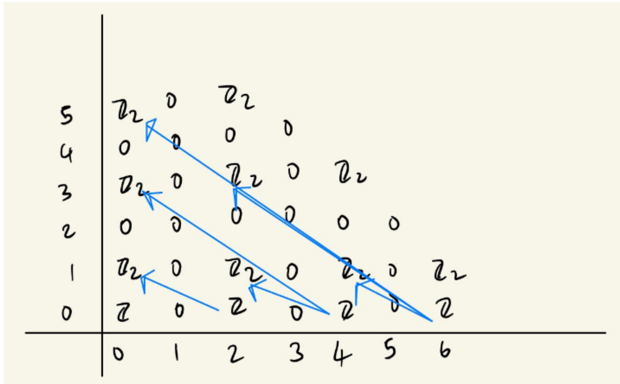
3. Under the isomorphisms $E_{p,q}^2 \approx H_p\left(B; H_q(F; G)\right)$ and $E_{p,q}'^2 \approx H_p\left(B'; H_q(F'; G)\right)$ the map f_*^2 corresponds to the map induced by the maps $B \to B'$ and $F \to F'$.

Consider a fibration $p : X \to B$ and a map to the identity fibration $B \to B$. We have the following commutative diagram:

$$\begin{array}{ccc} H_n(X; G) & \xrightarrow{p_*} & H_n(B; G) \\ \downarrow & & \downarrow{\scriptstyle =} \\ E_{n,0}^\infty(X) & \longrightarrow & E_{n,0}^\infty(B) \end{array}$$

This gives a factorization of p_* as the composition of the natrual surjection $H_n(X; G) \to E_{n,0}^\infty$ coming form the filtration in the first fibration, followed by the lower horizontal map, an injection. The latter map is the composition $E_{n,0}^\infty(X) \hookrightarrow E_{n,0}^2(X) \to E_{n,0}^2(B) = E_{n,0}^\infty(B)$ whose second map will be an isomorphism if the fiber F of the fibration $X \to B$ is path-connected. This factorization must be equivalent to the canonical factorization $H_n(X; G) \to \operatorname{Im} p_* \hookrightarrow H_n(B; G)$

Example 4. Consider the fibration $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$ inducing multiplication by 2 on π_2. Thus this has fiber $K(\mathbb{Z}_2, 1)$. In the E^2 page the differentials originating from above the 0^{th} row must have source or target 0, so must be trivial.



Every diff from $\mathbb{Z}$ in 0^{th} row to $\mathbb{Z}_2$ in upper row must be nontrivial. Since there is no $\mathbb{Z}_2$ in $H_*(K(\mathbb{Z},2);\mathbb{Z})$. The diff $\mathbb{Z} \to Z_2$ send generators to generators. Say $1 \mapsto 1$, when this happens $\mathbb{Z}_2$ is killed and source $\mathbb{Z}$ becomes $\ker(\mathbb{Z} \to \mathbb{Z}/2)$ which is $2\mathbb{Z} \subset \mathbb{Z}$. Note $2\mathbb{Z} \simeq \mathbb{Z}$. On the next page we have nontrivial diff $2\mathbb{Z} \simeq \mathbb{Z} \to \mathbb{Z}_2$. THis gives $4\mathbb{Z} \subset \mathbb{Z}$. This gives $E^\infty_{2n,0} = 2^n\mathbb{Z} \subset \mathbb{Z}$. So $p_* : H_{2n}(K(\mathbb{Z},2);\mathbb{Z}) \to E^\infty_{2n,0} \hookrightarrow H_{2n} = E^2_{2n,0}$. This gives that $Im(p_*) \leq H_{2n}(K(\mathbb{Z},2);\mathbb{Z})$ of index 2^n. So $p_* = \mu(2^n)$.