# Spectral sequences

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Our basic goal: Calculate $H^*$ where this is usually a graded $R$- module or $k$-algebra. We will mainly look at it in the topological setting. So $H^*$ for us would be homology or cohomology rings.
Long exact sequence coming from short exact sequence of (co)chain copmolexes in homology is a fundamental tool for computing (co)homology. One can consider filtered chain complexes coming from a filtration of a topological space. There is a generalization of a long exact sequence called spectral sequence that is a powerful tool for computing homology of a chain complex. We begin with looking at the basic definitions.

**Defintion.** *A spectral sequence is the data of*

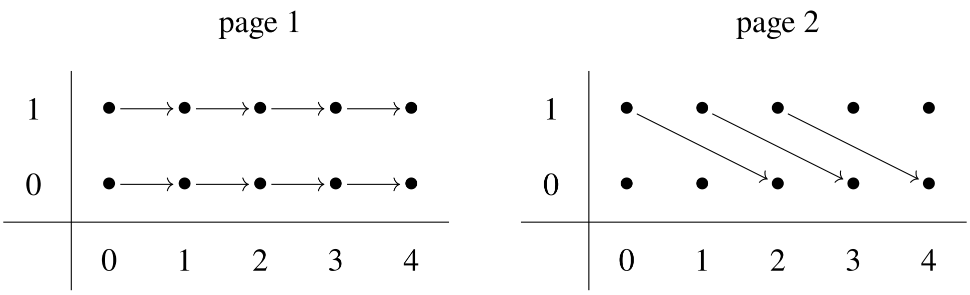
- *a sequence of bigraded objects* $\left\{E_r^{s,t}\right\}$ *with* $r \in \mathbb{N}$
- *differentials* $d_r : E_r \to E_r$ *that satisfy* $d_r^2 = 0$

*and of bidegree* $(-r, r-1)$*, in which case it is of homological type, or* $(r, 1-r)$*, in which case it is of cohomological type. In addition, we require*

$$E_{r+1}^{s,t} \cong \ker\left(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}\right) / \mathrm{im}\left(d_r : E^{p-r,q+r-1} \to E_r^{p,q}\right)$$

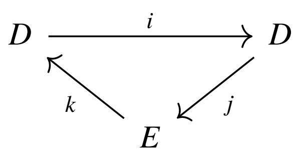
*in the case of cohomology, and the corresponding condition with degrees negated for homology.*

spectral sequences are imagined as a book with pages indexed by $r$ containing an integer lattice with $E_r^{*,*}$ at each point of the lattice. We will mostly consider sseq with $E_r^{p,q} - 0$ for $p, q < 0$. These are first quadrant spectral sequences. They usually look like:




The next question we would like to answer is "How do spectral sequences arise?"

Exact couples are a general source of sseq. An exact couple $C = \langle D, E; i, j, k\rangle$ where $E, D$ are modules, is a diagram:




where $\ker j = \mathrm{im}\, i, \ker = \mathrm{im}\, j, \ker i = \mathrm{Im}\, k$.

$d = jk : E \to E$ then $d^2 = 0$. Construct $C' = \langle D', E'; i', j', k'\rangle$ by

$D' = i(D), E' = H_*(E; d), i' = |i_{A'}, j'[ia] = [ja], k'[e] = [ke]$.

$C'$ is also a derived couple and simlarly we define $C^r$. $\{E^r, d^r\}$ gives terms of spectral sequence.

An unenrolled exact couple is a diagram of abelian groups of the form

$$\begin{array}{ccccccccc}
\dots \longrightarrow & D^{s+1,\cdot} & \xrightarrow{i} & D^{s,\cdot} & \longrightarrow & D^{s-1,\cdot} & \longrightarrow \dots \\
 & \downarrow & \nwarrow^{k} & \downarrow j & \nwarrow & \downarrow & \\
 & E^{s+1,\cdot} & & E^{s,\cdot} & & E^{s-1,\cdot} &
\end{array}$$

where each triangle is a rolled up long exact sequence of the form

$$\cdots \to D^{s+1,t+1} \xrightarrow{i} D^{s,t} \to E^{s,t} \to D^{s+1,t} \to \cdots$$

It can be completely unenrolled to get

$$\begin{array}{ccccccccccc}
\longrightarrow E_{s,t+1} & \longrightarrow & A_{s-1,t+1} & \longrightarrow & E_{s-1,t+1} & \longrightarrow & A_{s-2,t+1} & \longrightarrow & E_{s-2,t+1} & \longrightarrow \\
 & & \downarrow & & & & \downarrow & & & \\
\longrightarrow E_{s+1,t} & \longrightarrow & A_{s,t} & \longrightarrow & E_{s,t} & \longrightarrow & A_{s-1,t} & \longrightarrow & E_{s-1,t} & \longrightarrow \\
 & & \downarrow & & & & \downarrow & & & \\
\longrightarrow E_{s+2,t-1} & \longrightarrow & A_{s+1,t-1} & \longrightarrow & E_{s+1,t-1} & \longrightarrow & A_{s,t-1} & \longrightarrow & E_{s,t-1} & \longrightarrow
\end{array}$$

We can write $D = \oplus D^{*,*}$ and $E = \oplus E^{*,*}$ to get an exact couple

$$\begin{array}{ccc}
D & \xrightarrow{i} & D \\
 & \nwarrow^{k} & \downarrow j \\
 & & E
\end{array}$$

Filtered chain complexes give rise to exact couples and hence spectral seqeunces.

Let $A = \mathbb{Z}$ graded complex of modules. Suppose we have an increasing filtration

$$\cdots \subset F_{p-1}A_* \subset F_pA_* \subset \cdots \subset A_*$$

Let $\mathrm{Assoc}(A) = E^0_{p,q}(A) = \left(\frac{F_pA}{F_{p-1}A}\right)_{p+q}$

Suppose $d$ is a differential and filtration on $A$ respects the differential then

$$F_pH_*(A) = \mathrm{Im}(H_*(F_pA) \xrightarrow{i_*} H_*(A))$$

So $E^0H_*(A)$ is also defined. We have short exact sequence of chain complexes

$$0 \to F_{p-1} \to F_pA \to E^0_{p,q}(A) \to 0$$

indcuing a long exact seuqence

$$\cdots H_n(F_{p-1}A_*) \to H_n(F_pA_*) \to H_n(E^0_pA) \to H_{n-1}(F_{p-1}A) \to \cdots$$

Here we have $D^*_{p,q} = H_{p+q}(F_pA)$ and $E^*_{p,q} = H_{p+q}(E^0_p(A))$

**Convergence**

We know a filtration $F^*$ of $H^*$ can be collapsed into associated graded vector space, defined by $E_0^p(H^*) = F^pH^*/F^{p+1}H^*$. In the case of a locally finite graded vector space (i.e., $H^n$ is finite dimensional for each $n$ ), $H^*$ can be recovered up to isomorphism from the associated graded vector space by taking direct sums, i.e.,

$$H^* \cong \bigoplus_{p=0}^{\infty} E_0^p(H^*)$$

For an arbitrary graded module $H^*$ this might be isomorphic up to an extension.

Because $H^*$ may not be easily computed, we can take as a first approximation to $H^*$ the associated graded vector space to some filtration of $H^*$. This is the target of a spectral sequence.

If the terms $E_r^{*,*}$ stabilize, i.e. for large $r$, $E_r = E_{r+1} = \cdots$ then we denote the common value by $E_\infty$.

A spectral sequence $\{E_r, d_r\}$ is said to converge to a graded module $G^*$ if there exists a filtration $F$ on $G^*$ such that

$$E_\infty^{p,q} \simeq \left(\frac{F_pG}{F_{p-1}G}\right)_{p+q}$$

We denote this by

$$E_r^{*,*} \implies G_*$$

Most common results look like this:

**Theorem.** *There exists a spectral sequence $\{E_r^{**}, d_r\}$ with*

$$E_2^{**} \simeq \text{“}something\ computable\text{”}$$

*and converging to $H^*$- something computable.*

We consider the fibration $F \to X \to B$. We know that fibers over a path component are homotopy equivalent and we have the following action on fiber

$$L_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$$

for a path $\gamma : I \to B$. The map $\gamma \mapsto L_\gamma$ induces an action of $\pi_1(B)$ on $H_*(F)$. We are interested in trivial action, so $L_{\gamma^*} = \mathrm{id}\,\forall\gamma$.

**Theorem 1** (**Serre spectral sequence**)**.** *Let $F \to X \to B$ be a fibration with $B$ path connected. If $\pi_1(B)$ acts trivially on $H_*(F)$ then there exists a spectral sequence $\{E^r_{p,q}, d_r\}$ with*

$$E^2_{p,q} \simeq H_p(B; H_q(F))$$

*and converging to $H_*(X)$, i.e.*

$$E^\infty_{p,n-p} \simeq \frac{F_n^p}{F_n^{p-1}}$$

*for a filtration $0 \subset F_n^0 \subset \cdots \subset F_n^n = H_n(X)$.*

**Example 2.** [**Homology of** $K(\mathbb{Z}, 2)$] Consider the pathspace fibration $F \to P \to B$ when $B = K(\mathbb{Z}, 2)$. $P$ is contractible so $F = K(\mathbb{Z}, 1)$.

$$H_i(F; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

So we have

$$E^2_{p,q} = \begin{cases} 0 & q > 2 \\ H_p(B) & q = 1, 2 \end{cases}$$

| | | | | | |
|---|---|---|---|---|---|
| 1 | $\mathbb{Z}$ | $H_1(B)$ | $H_2(B)$ | $H_3(B)$ | $\cdots$ |
| 0 | $\mathbb{Z}$ | $H_1(B)$ | $H_2(B)$ | $H_3(B)$ | $\cdots$ |
| | 0 | 1 | 2 | 3 | 4 |

$d^3, d^4$ are all 0. So $E^3 = \cdots = E^\infty$. $P \simeq \star$ so $E^\infty$ page has one $\mathbb{Z}$ at $(0, 0)$. Hence $d^2$ must be isomorphism except $d^2 : E^2_{0,0} \to 0$.

This gives us $H_1(B) = 0$ and $H_2(B) \simeq \mathbb{Z}$.

There is an analogous **Serre spectral sequence in cohomology**.

**Theorem 3.** *For a fibration $F \to X \to B$ with $B$ path connected and $\pi_1(B)$ acting trivially on $H^*(B)$ there exists a spectral sequence*

$$E_2^{p,q} \simeq H^p(B; H^q(F))$$

*converging to $H^*(X)$, i.e.*

$$E_\infty^{p,n-p} \simeq \frac{F^n_p}{F^n_{p+1}}$$

*for a filtration $0 \subset F^n_n \subset \cdots \subset F^n_0 = H^n(X)$.*

The Serre spectral sequence admits a bilinear product

$$E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t} \quad \forall r \tag{1}$$

satisfying

- Each $d_r$ is a derivation satisfying

  $$d(xy) = (dx)y + (-1)^{p+q} x dy$$

  So the product in eq. (1) induces a product $E_{r+1} \times E_{r+1} \to E_{r+1}$.

- The product $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$ is $(-1)^{qs}$ times the standard cup product $H^p(B; H^q(F)) \times H^s(B; H^t(F)) \to H^{p+s}(B; H^{q+t}(F))$

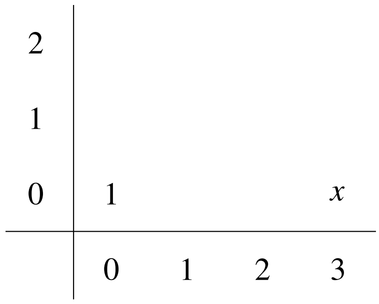
- The cup product in $H^*(X)$ restricts to maps $F^m_p \times F^n_s \to F^{m+n}_{p+s}$. These induce quotient maps $F^m_p / F^m_{p+1} \times F^n_s / F^n_{s+1} \to F^{m+n}_{p+s} / F^{m+n}_{p+s+1}$ that coincide with the products $E_\infty^{p,m-p} \times E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$.

**Example 4.** We would like to compute $H^*(\Omega S^n; \mathbb{Q})$. We will do a simpler case of $n = 3$, the generalization follows as a simple case-work for odd and even cases.

We have the fibration $\Omega S^3 \to PS^3 \to S^3$ that satisfies the hypothesis of Serre spectral seqeunce.

$$E^2_{p,q} = H^p(S^3; H^q(\Omega S^3; \mathbb{Q})) \implies H^{p+q}(PS^3; \mathbb{Q}) = \begin{cases} \mathbb{Q} & p + q = 1 \\ 0 & \text{otherwise} \end{cases}$$

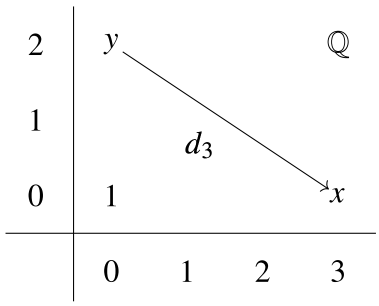
$E^2$ page: $H^*(S^3; \mathbb{Q}) = \frac{\mathbb{Q}[x]}{x^2}$ where $|x| = 3$ and $H^0(\Omega S^3; \mathbb{Q}) = \mathbb{Q}$.




$E^{3,0}_\infty$ has to be 0. So $E^{3,0}_2$ has to vanish. This means there exists a non zero differential map in or out of $E^{3,0}_2$. This gives the only possibility $d_2 : E^{1,1}_2 \to E^{3,0}_2$, but $E^{1,1} = 0$. $d_r$ for $r \geq 4$ comes from 4th quadratn and are 0. So $d_3 : E^{0,2}_3 \to E^{3,0}_3$ is non-zero and is an isomorphism. So there exists $y \in E^{0,2}_2$ such that $d_3 y = x$ ( There can't be more than one generator at $E^{0,2}$)

$E^{0,2}_2 = H^2(\Omega S^3; \mathbb{Q}) \simeq \mathbb{Q}[y]$. This implies $E^{3,2}_2 = H^3(S^3; \mathbb{Q}) \simeq \mathbb{Q}$

So we have




$E^{0,1}_2 = 0$ because there are no differential that hit something non-zero or map from something non-zero to it. S0 $E^{0,1}_2 = 0$.

Multiplicative structure tells us that $E^{3,2}_2 = y \cdot x$.

$\mathbb{Q}[yx]$ cannot remain till $E_\infty$ page. So

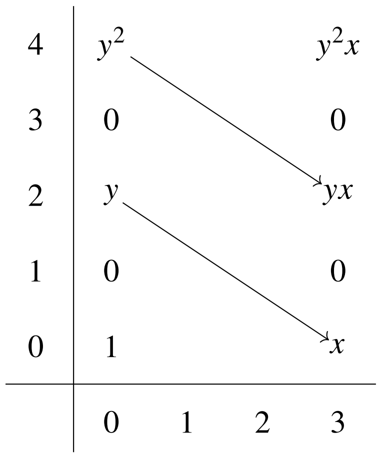
$$d_3 : E^{0,4}_2 \to \mathbb{Q}[yx]$$

is non-trivial and an isomorphism. $d_3 y = x$, $d_3 y^2 = 2xy$ so the above map is multiplication by 2.

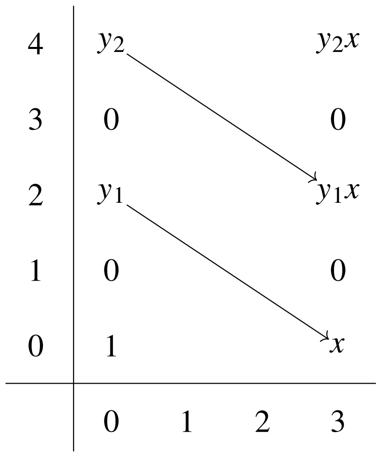
$$E^{0,4}_2 = \mathbb{Q}[y^2]$$

For $y^2x$ to vansih, it is killed by $y^3 \in E_2^{0,0}$ and $d_3 = \mu(3)$.

$$H^*(\Omega S^3; \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{even} \\ 0 & \text{odd} \end{cases}$$




With $\mathbb{Z}$ coefficients we have a sequence of generators $y_1, y_2, \ldots$




where the following relations hold true: $y_1^2 = c_1 y_2$ for some $c_1 \in \mathbb{Z}$.
$d_3(y_2) = y_1 x$ and $d_3(y_1^2) = 2y_1 x$. This implies $y_1^2 = 2y_2$, since $d_3$ is an isomorphism. Similarly $y_1^n = n! y_n$. We get $H^*(\Omega S^3; \mathbb{Q})$ to be a divided power algebra.