On real motivic $\mathcal{Z}^{\mathbb{R}}$

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Abstract

We show that the realization of $\mathcal{A}^{\mathbb{R}}$-module $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$ is of chromatic type $(2,2)$. We also describe the structure of the $\mathcal{A}^{\mathbb{R}}$-module $\mathcal{A}^{\mathbb{R}}(2)$.

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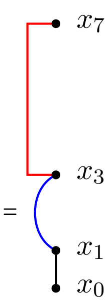
The $\mathcal{A}^{\mathbb{R}}$-module $\mathcal{A}^{\mathbb{R}}(2)$ is realized using the following construction. We follow [BGL, Cor 5.3]. There exists a complex $\mathcal{K} \in \mathrm{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ such that $\mathrm{H}^{\star,*}_{\mathbb{R}}(\mathcal{K})$ is $\mathbb{M}^{\mathbb{R}}_2$-free on four generators x_0, x_1, x_3 and x_7, such that $\mathrm{Sq}^{i+1}(x_i) = x_{2i+1}$ for $i \in \{0,1,3\}$.

$H^{\star,*}_{\mathbb{R}}(\mathcal{K}) =$



Let $e \in \mathbb{Z}_{(2)}[\Sigma_6]$ be the idempotent corresponding to the Young tableaux

1	2	3
4	5	
6		

We define

$$\mathcal{A}^{\mathbb{R}}_2 \coloneqq \Sigma^{-5,-1} e(\mathcal{K}^{\wedge 6})$$

This realizes $\mathcal{A}^{\mathbb{R}}(2)$, i.e., $\mathrm{H}^{\star,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2) = \mathcal{A}^{\mathbb{R}}(2)$ ([BGL, Theorem 5.6]).

Let $\widetilde{Q}^{\mathbb{R}}_2 = [\mathrm{Sq}^4, Q^{\mathbb{R}}_1]$. Unlike the $\mathbb{C}$-motivic or the classical case, the dual to τ_2 is given by

$$Q^{\mathbb{R}}_2 = \widetilde{Q}^{\mathbb{R}}_2 + \rho \mathrm{Sq}^5 \mathrm{Sq}^1$$

$\widetilde{Q}_2^{\mathbb{R}}$ still generates an exterior algebra, $\bigwedge(\widetilde{Q}_2)$. We define the left $\mathcal{A}^{\mathbb{R}}(2)$-module

$$\widetilde{B}^{\mathbb{R}}(2) \coloneqq \mathcal{A}^{\mathbb{R}}(2) \otimes_{\bigwedge(\widetilde{Q}_2^{\mathbb{R}})} M_2^{\mathbb{R}}$$

There is a short exact sequence of $\mathcal{A}^{\mathbb{R}}(2)$-modules

$$0 \to \Sigma^{7,3}\widetilde{\mathcal{B}}^{\mathbb{R}}(2) \hookrightarrow \mathcal{A}^{\mathbb{R}}(2) \twoheadrightarrow \widetilde{\mathcal{B}}^{\mathbb{R}}(2) \to 0 \tag{1}$$

which can be updated to an exact sequence of $\mathcal{A}^{\mathbb{R}}$-modules ([BGL, Lemma 5.9]). Let $\mathcal{Z}_{\mathbb{R}} \in \mathrm{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ be the realization of $\widetilde{B}^{\mathbb{R}}(2)$.

1 Chromatic type of $\mathcal{Z}^{\mathbb{R}}$

Definition 1.1. A spectrum $X \in \mathrm{Ho}(Sp^{C_2})_{2,\mathrm{fin}}$ is of type (m,n) if $\Phi^e(X)$ is of type m and $\Phi^{C_2}(X)$ is of type m.

The type of a spectrum $X \in \mathrm{Ho}(Sp^{\mathbb{R}})_{2,\mathrm{fin}}$ is defined to be the type of the spectrum $\beta(X)$, where β is the Betti realization functor

$$\beta : \mathrm{Ho}(Sp^{\mathbb{R}})_{2,\mathrm{fin}} \to \mathrm{Ho}(Sp^{C_2})_{2,\mathrm{fin}}$$

Proposition 1.2. The complex $\mathcal{Z}^{\mathbb{R}}$ is type $(2,2)$.

Proof. The underlying spectrum of $\beta(\mathcal{Z}^{\mathbb{R}})$ is $\mathcal{Z} \in \mathrm{Ho}(Sp)_{2,\mathrm{fin}}$, which is type 2 ([BE]).

Let $\mathbf{B} = \beta(\mathcal{Z}^{\mathbb{R}})$. To show that $\Phi^{C_2}(\mathbf{B})$ is type 2, we show that $K(1)_*(\Phi^{C_2}\mathbf{B})$ is trivial and $K(2)_*(\Phi^{C_2}\mathbf{B})$ is nonzero. We will show in section 4 that Q_1 acts freely on $H^*\Phi^{C_2}\mathbf{B}$. It follows that $\mathrm{Ext}^{s,f}_{E(Q_1)}(\Phi^{C_2}(\mathbf{B}),\mathbb{F}_2) = 0$ for $f > 0$. All the elements in $\mathrm{Ext}^{s,0}$ are v_1-torsion, since $\mathrm{Ext}^{s,f} = 0$ for $f > 0$. It follows that $k(1)_*\Phi^{C_2}\mathbf{B}$ is v_1-torsion and that $K(1)_*\Phi^{C_2}\mathbf{B} = 0$. We will show in section 4 that Q_2 acts trivially on $H^*\Phi^{C_2}\mathbf{B}$. We show below that there are no v_2 torsion elements in $k(2)_*\Phi^{C_2}\mathbf{B}$. It follows that $K(2)_*(\Phi^{C_2}(\mathbf{B}))$ is non-zero.

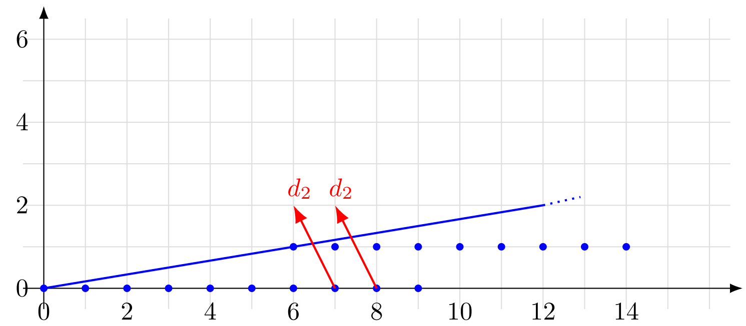
We describe the structure of $\mathcal{A}^{\mathbb{R}}(2)$ in section 3. We also determine the Sq^8 actions on $\mathcal{A}^{\mathbb{R}}(2)$s using the Cartan formula on the basis elements $x_{i_1} \otimes \cdots \otimes x_{i_6}$ of $\mathcal{K}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}} 6}$. The Sq^8 actions on elements in $\mathcal{A}^{\mathbb{R}}(2)$ are recorded up until stem 10 in section 3.

([BGL]) The fixed points functor takes the Steenrod actions $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \mathrm{Sq}^8$ to $0, \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4$ respectively. We can now determine Q_i actions for $i = 1,2$ on $\Phi^{C_2}(\mathbf{B})$. These are recorded in section 4.

Since the action of Q_2 is trivial,

$$\begin{aligned}
\mathrm{Ext}_{E(Q_2)}(\Phi^{C_2}(\mathbf{B}),\mathbb{F}_2) &\cong \mathrm{Ext}^{*,*}_{E(Q_2)}(\bigoplus_c \Sigma^{|c|}\mathbb{F}_2, \mathbb{F}_2) \\
&\cong \bigoplus_c \mathrm{Ext}^{*,*+|c|}_{E(Q_2)}(\mathbb{F}_2,\mathbb{F}_2) \\
&\cong \bigoplus_c \mathbb{F}_2[v_2]
\end{aligned}$$

where $|v_2|$ has bidegree $(6,1)$. The E_2-page of the corresponding Adams spectral sequence, is drawn below. There is no non-trivial target for any differentials. Hence there are no v_2-torsion elements in $k(2)_*(\Phi^{C_2}(\mathbf{B}))$, and thus $K(2)_*\Phi^{C_2}\mathbf{B}$ is non-zero.



$\square$

In [BC, Thm 6.39], the authors prove a limited version of the Hopkins-Smith periodicity theorem in the equivariant setting. Applying that to our case we can say that the complex $\mathcal{Z}^{\mathbb{R}}$ should admit a $v_{(-1,2)}$ self-map.

2 Different Sq^8 actions

We will show that the Sq^8 action is different on the two copies of $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$ in $\mathcal{A}_2^{\mathbb{R}}$. There is a copy of $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$ generated by $\widetilde{Q}_2$ written $\Sigma^{7,3}\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$ and the quotient copy $\mathcal{A}_2^{\mathbb{R}}/\mathcal{A}_2^{\mathbb{R}}\cdot\widetilde{Q}_2$ denoted $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$.

The expressions for Sq^8 on Sq^0, Sq^1 and $\mathrm{Sq}^0\widetilde{Q}_2$, $\mathrm{Sq}^1\widetilde{Q}_2$ match for $\Sigma^{7,3}\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$ and $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$.

In $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$,

$$\mathrm{Sq}^8(\mathrm{Sq}^2) = \mathrm{Sq}^4\mathrm{Sq}^6 + \tau(\mathrm{Sq}^6\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^7\mathrm{Sq}^3)$$

but in $\Sigma^{7,3}\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$,

$$\mathrm{Sq}^8(\mathrm{Sq}^2\widetilde{Q}_2) = (\mathrm{Sq}^4\mathrm{Sq}^6 + \tau(\mathrm{Sq}^6\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^7\mathrm{Sq}^3) + \tau\mathrm{Sq}^2\mathrm{Sq}^7\mathrm{Sq}^1)\widetilde{Q}_2$$

The term $\tau\mathrm{Sq}^2\mathrm{Sq}^7\mathrm{Sq}^1$ is not an element of $\Sigma^{7,3}\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$.

Also,

$$\mathrm{Sq}^8(\mathrm{Sq}^3\widetilde{Q}_2) = (\mathrm{Sq}^4\mathrm{Sq}^7 + \mathrm{Sq}^2\mathrm{Sq}^7\mathrm{Sq}^2 + \tau\mathrm{Sq}^7\mathrm{Sq}^3\mathrm{Sq}^1 + \rho\mathrm{Sq}^2\mathrm{Sq}^7\mathrm{Sq}^1)\widetilde{Q}_2$$

and

$$\mathrm{Sq}^8(\mathrm{Sq}^2\mathrm{Sq}^1\widetilde{Q}_2) = \mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^7\mathrm{Sq}^2 + \rho\mathrm{Sq}^2\mathrm{Sq}^7\mathrm{Sq}^1$$

with the term $\rho\mathrm{Sq}^2\mathrm{Sq}^7\mathrm{Sq}^1$ not in the expression of the corresponding terms in $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$.

The expressions for Sq^8 on $\mathrm{Sq}^3\mathrm{Sq}^1$ and Sq^4 are the same for the two copies of $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$.

We note down one more difference in Sq^8 action

$$\mathrm{Sq}^8(\mathrm{Sq}^4\mathrm{Sq}^1\widetilde{Q}_2) = (\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^1 + \rho\mathrm{Sq}^6\mathrm{Sq}^5\mathrm{Sq}^1)\widetilde{Q}_2$$

Upon quotienting by $\mathcal{A}_2^{\mathbb{R}}\cdot\widetilde{Q}_2$, the difference is $\rho\mathrm{Sq}^3\mathrm{Sq}^7\mathrm{Sq}^2$.

This shows that in 1, the Sq^8 actions on $\Sigma^{7,3}\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$ and $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$ are different. Since the two $\mathcal{A}^{\mathbb{R}}$-module structures on $\widetilde{B}_2^{\mathbb{R}}$ are different, the element v_2 represented by the extension 1 in $\mathrm{Ext}^{6,1}_{\mathcal{A}^{\mathbb{R}}}(\widetilde{B}_2^{\mathbb{R}}, \widetilde{B}_2^{\mathbb{R}})$ will not give rise to a self-map on $\mathcal{Z}^{\mathbb{R}}$

3 $\mathcal{A}^{\mathbb{R}}(2)$ module structure

Here a string of numbers, say a, b, c repesent the monomial in Steenrod algebra $\mathrm{Sq}^a\mathrm{Sq}^b\mathrm{Sq}^c$.

Stem 1-5

	Sq^1	Sq^2	Sq^4
$0 = Sq^0$	1	2	4
1	0	$2,1$	$4,1$
2	$2,1$	$\tau 3,1$	$4,2$
$2,1$	$3,1$	0	$4,2,1$
3	0	$5 + 4,1 + \tau 3,1$	$5,2$
4	5	$6 + \tau 5,1$	$\tau 7,1 + 6,2$
$3,1$	0	$5,1$	$5,2,1$
5	0	$6,1 + \rho 5,1$	$9 + 8,1 + \rho 7,1 + 7,2$
$4,1$	$5,1$	$6,1$	$6,2,1$

Stem 6

	Sq^1	Sq^2	Sq^4
$4,2$	$5,2$	$6,2$	$\tau(7,3 + 6,3,1)$
6	7	$\tau 7,1$	$10 + 8,2$
$5,1$	0	0	$9,1 + 7,2,1$

Stem 7

	Sq^1	Sq^2	Sq^4
$5,2$	0	$6,3$	$9,2+8,3+\rho 7,3+\tau 7,3,1+\rho 6,3,1$
7	0	$9+8,1$	$11+9,2$
$6,1$	$7,1$	0	$10,1+8,2,1$
$4,2,1$	$5,2,1$	$6,2,1$	$\tau 7,3,1$

Stem 8

	Sq^1	Sq^2	Sq^4
$5,2,1$	0	$6,3,1$	$9,2,1+8,3,1+\rho\cdot 7,3,1$
$6,2$	$7,2$	$\tau\cdot 7,3$	$10,2+\tau\cdot 8,3,1+\rho\tau\cdot 7,3,1$
$7,1$	0	$9,1$	$11,1+\tau\cdot 9,2,1$

Stem 9

	Sq^1	Sq^2	Sq^4
$6,3$	$7,3$	0	$10,3+9,4+8,4,1+\rho 8,3,1$
$9+8,1$	$9,1$	$\rho 9,1$	$\rho 11,1+11,2+10,2,1$
$7,2$	0	$9,2+8,3+\rho 7,3$	$11,2+\tau 9,3,1+\rho 8,3,1+\rho^2 7,3,1$
$6,2,1$	$7,2,1$	$\tau\cdot 7,3,1$	$10,2,1$

Stem 10

	Sq^1	Sq^2	Sq^4
$6,3,1$	$7,3,1$	0	$10,3,1+9,4,1$
$7,3$	0	$9,3$	$11,3+9,4,1+\rho 9,3,1$
$9,1$	0	0	$11,2,1$
$7,2,1$	0	$9,2,1+8,3,1$	$11,2,1$
$10+8,2$	$11+9,2$	$10,2+\tau 9,3+\tau 11,1$	$\tau 11,3+\tau 10,3,1+\rho\tau 9,3,1$

Stem 11

	Sq^1	Sq^2	Sq^4
$9,2+8,3$	$9,3$	$\rho 9,3$	$\rho 11,3+\tau 11,3,1+\rho^2 9,3,1+$ $10,5+10,4,1$
$7,3,1$	0	$9,3,1$	$11,3,1$
$11+9,2$	0	$13+12,1+\rho 11,1+10,3+\rho 9,3$	$13,2+12,3+\rho 11,3+$ $\tau 11,3,1+\rho 10,3,1+\rho^2 9,3,1$
$10,1+8,2,1$	$11,1+9,2,1$	$10,2,1+\tau 9,3,1$	$\tau 11,3,1$

Stem 12

	Sq^1	Sq^2	Sq^4
$9,2,1+8,3,1$	$\rho 9,3,1$	$9,3,1$	$\rho 11,3,1+10,5,1$
$10,2+\tau 8,3,1$	$\tau(11,3+10,3,1+\rho 9,3,1)$	$\rho 8,3,1+\tau 9,3,1+11,2$	$13,2,1+12,3,1+\rho 11,3,1$
$11,1+9,2,1$	0	$13,1+10,3,1+\rho 9,3,1$	$13,2,1+12,3,1+\rho 11,3,1$
$9,3$	0	0	$11,5+11,4,1+\rho 11,3,1$

Stem 13

	Sq^1	Sq^2	Sq^4
$10,3+9,4+8,4,1+\rho 8,3,1$	$11,3+9,4,1+\rho 9,3,1$	$10,5+10,4,1+\rho 10,3,1$	$\tau 11,5,1$
$13+12,1+10,3+\rho 11,1$	$13,1+11,3$	0	$17+16,1+\rho 15,1+$ $15,2+14,2,1+12,5+12,4,1$ $+\rho 12,3,1+\rho 13,2,1$
$9,3,1$	0	0	$11,5,1$
$11,2+\tau 9,3,1+\rho 8,3,1$	0	0	$11,5,1$
$10,2,1$	$11,2,1$	$\tau 11,3,1$	0

stem 14

	Sq^1	Sq^2	Sq^4
$11,3+9,4,1$	0	$13,3+10,5,1$	$13,5+13,4,1+\rho 13,3,1+12,5,1$
$10,3,1+9,4,1$	$11,3,1$	$10,5,1$	0
$11,2,1$	0	$13,2,1+12,3,1+\rho 11,3,1$	0
$13,1+10,3,1$	$11,3,1$	0	$17,1+15,2,1+12,5,1$

stem 15

	Sq^1	Sq^2	Sq^4
$11,3,1$	0	$13,3,1$	$13,5,1$
$13,2+12,3$	$13,3$	$\rho 13,3$	$17,2+16,3+\rho 15,3+\tau 15,3,1+$ $\rho^2 13,3,1+14,5+14,4,1$
$10,5+10,4,1+\rho 11,3$	$11,5+11,4,1$	$\tau 11,5,1+\rho 13,3$	$\rho 12,5,1+\rho 13,5+\rho 13,4,1+\rho^2 13,3,1$

stem 16

	Sq^1	Sq^2	Sq^4
$13,3+10,5,1$	$11,5,1$	0	$17,3+15,5+15,4,1+\rho 15,3,1$
$11,5+11,4,1$	0	$13,5+13,4,1+12,5,1+\rho 11,5,1$	$\rho 13,5,1$
$10,5,1$	$11,5,1$	0	0
$13,2,1+12,3,1$	$13,3,1$	$\rho 13,3,1$	$17,2,1+16,3,1+\rho 15,3,1+14,5,1$

stem 17

	Sq^1	Sq^2	Sq^4
$11,5,1$	0	$13,5,1$	0
$13,3,1$	0	0	$17,3,1+15,5,1$
$\alpha_{17} = 17+16,1+\rho 15,1+$ $15,2+14,2,1$ $12,5+12,4,1$	$17,1+15,2,1+$ $13,5+13,4,1$	$17,2+16,3+$ $\rho 15,3+\tau 15,3,1+$ $14,5+14,4,1+\tau 13,5,1$	$\rho 17,2,1+\tau 17,3,1+\rho 16,3,1+$ $\rho^2 15,3,1+\rho 14,5,1$ $+\tau 15,5,1$

stem 18

	Sq^1	Sq^2	Sq^4
$13,5+13,4,1+12,5,1$	$13,5,1$	$\rho 13,5,1$	$17,5+17,4,1+16,5,1$
$17,1+15,2,1+12,5,1$	$13,5,1$	$17,2,1+16,3,1+\rho 15,3,1+14,5,1$	0

Stem 19-23

	Sq^1	Sq^2	Sq^4
$17,2+16,3+\rho 15,3+$ $\tau 15,3,1+14,5+14,4,1$	$17,3+15,5$ $+15,4,1+\rho 15,3,1$	$\tau 17,3,1+\tau 15,5,1$	$17,5,1$
$13,5,1$	0	0	$17,5,1$
$17,3+15,5+15,4,1+\rho 15,3,1$	0	$17,5+17,4,1$ $+16,5,1+\rho 15,5,1+\rho 17,3,1$	0
$17,2,1+16,3,1+\rho 15,3,1+14,5,1$	$17,3,1+15,5,1$	0	0
$17,3,1+15,5,1$	0	$17,5,1$	0
$17,5+17,4,1+16,5,1$	$17,5,1$	0	0
$17,5,1$	0	0	0

Sq^8 action on elements in $\mathcal{A}_2^{\mathbb{R}}$ until stem 10

Sq^0	$6,2+5,2,1+\tau 7,1$
1	$2,7+7,2$
2	$4,6+\tau(6,3,1+7,3)$
3	$4,7+2,7,2+\tau 7,3,1$
$2,1$	$4,6,1+2,7,2$
4	$\tau(2,7,3+5,6,1+2,7,2,1)$
$3,1$	$2,7,3+5,6,1+2,7,2,1$
5	$\tau 2,7,3,1+\rho(2,7,3+5,6,1+2,7,2,1)$
$4,1$	$4,6,3+2,1,4,6$
$4,2$	0
$5,1$	$4,6,3,1+2,1,4,6,1$
6	$\tau(4,6,3,1+5,6,3)$
$5,2$	0
$6,1$	$4,4,7+2,4,6,3$
7	$4,1,4,6$
$4,2,1$	0
$6,2$	0
$5,2,1$	0
$7,1$	$2,5,6,2,1+5,2,7,2+\rho 5,7,3+\rho^2 3,7,3,1$
$6,3$	$3,4,6,3,1$
$7,2$	$\tau 3,5,6,2,1$
$6,2,1$	0
$9+8,1$	$4,2,1,4,6+\rho(4,2,7,3+5,2,7,2+\rho 5,7,3+\rho^2 3,7,3,1)+\tau 3,5,6,2,1$
$10+8,2$	0
$6,3,1$	0
$7,2,1$	0
$7,3$	$2(5,2,7,2+\rho 5,7,3+\rho 5,6,3,1+\rho^2 3,7,3,1)+3,5,6,2,1$
$9,1$	$4,2,1,4,6+\rho 4,2,7,3+\rho(2,5,6,2,1+\rho 5,6,3,1)+\rho(3,5,6,2,1)$

Generators of $\mathcal{A}^{\mathbb{R}}(2)$ in terms of $\mathcal{A}^{\mathbb{R}}(2)$-module structure

We note the generators in stem 9 through 16 (top degree of $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$) entirely in terms of the $\mathcal{A}^{\mathbb{R}}(2)$-module structure.

$9+8,1$	$2,7$
$9,1$	$2,7,1$
$10+8,2$	$4,6$
$9,2+8,3$	$2,7,2+\rho 7,3$
$11+8,2$	$5,6$
$10,1+8,2,1$	$4,6,1$
$9,2,1+8,3,1$	$2,7,2,1$
$10,2{+}\tau 8,3,1$	$4,6,2+\rho\tau 7,3,1$
$11,1+9,2,1$	$5,6,1$
$9,3$	$3,7,2$
$10,3+9,4+8,4,1+\rho 8,3,1$	$4,6,3$
$13+12,1+10,3+\rho 11,1$	$2,1,4,6+\rho 2,7,3$
$9,3,1$	$2,7,3,1$
$11,2+\tau 9,3,1+\rho 8,3,1$	$4,7,2+\rho^2 7,3,1$
$10,2,1$	$4,6,2,1$
$11,3+9,4,1$	$4,7,3+\rho 2,7,3,1$
$10,3,1+9,4,1$	$4,6,3,1$
$11,2,1$	$5,6,2,1$
$13,1+10,3,1$	$2,5,6,1+\rho 2,7,3,1$
$11,3,1$	$5,6,3,1$
$13,2+12,3$	$4,5,6+\rho(4,7,3+4,6,3,1)+\tau 5,6,3,1$
$10,5+10,4,1+\rho 11,3$	$4,2,7,2+\rho 4,7,3+\tau 5,6,3,1+\rho^2 2,7,3,1$
$13,3+10,5,1$	$2,4,7,3+\rho 2,2,7,3,1$
$11,5+11,4,1$	$5,2,7,2+\rho 5,7,3+\rho 5,6,3,1+\rho^2 3,7,3,1$
$10,5,1$	$2,4,6,3,1$
$13,2,1+12,3,1$	$2,5,6,2,1+\rho 5,6,3,1$

4 Q_i actions on $\Phi^{C_2}(\mathrm{B})$

$Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$ action on $\Phi^{C_2}(\mathrm{B})$:

0	$4,2+6$
1	$6,1+4,2,1$
2	$6,2$
$2,1$	$6,2,1$
3	$5,2,1+7,2$
4	$4,6$
$3,1$	$7,2,1$
5	$5,6+7,2,1$
$4,2$	$4,6,2$
$4,1$	$2,7,2+7,3+5,6+7,2,1$
6	$4,6,2$
$5,1$	0
$5,2$	$3,7,2+4,7,2+7,3,1$
$6,2$	$4,7,2+7,3,1$
$6,1$	$4,6,2,1$
$4,2,1$	$4,6,2,1$
$5,2,1$	$5,6,2,1$
$7,2$	$4,5,6+4,7,3+4,6,3,1$
$6,3$	0
$6,2,1$	0
$4,6$	0
$6,3,1$	$2,4,6,3,1$
$7,2,1$	0
$2,7,2+7,3$	0
$4,6,2$	0
$5,6$	0

Q_2 action on $\Phi^{C_2}\mathbf{B}$: When we calculate the action of $Q_2 = \mathrm{Sq}^4\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^4 + \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^4$ on $\Phi^{C_2}\mathbf{B}$ we see that it is trivial or 0 for all elements.

5 Extending theorem 2.3 from [BE]

Theorem 5.1. Let B_2 denote any $\mathcal{A}$-module which restricts as an $\mathcal{A}(2)$-module to $B(2)$. Then there exists an element $\overline{v}_2 \in \mathrm{Ext}^{6,1}_{\mathcal{A}}(B_2, B_2)$ which maps to $\tilde{v}_2$ under the map

$$k : \mathrm{Ext}^{s,f}_{A}(B_2, B_2) \to \mathrm{Ext}^{s,f}_{A(2)}(B(2), B(2))$$

Proof. We will prove this with the help of spectral seqeunces.

The *algebraic Atiyah-Hirzebruch* spectral sequence takes the form

$$\bigoplus_{\mathcal{G}} \mathrm{Ext}^{s,f}_{A}(B_2, \mathbb{F}_2) \implies \mathrm{Ext}^{s,f}_{A}(B_2, B_2)$$

where $\mathcal{G}$ is an $\mathbb{F}_2$-basis of B_2.

We find $\mathrm{Ext}^{s,f}_{A}(B_2, \mathbb{F}_2)$ we use *May*'s spectral sequence

$$E^{\mathrm{May}}_{1,B_2} \coloneqq \mathrm{Ext}^{s,t}_{\mathrm{gr}(A)}(\mathrm{gr}(B_2), \mathbb{F}_2) \implies \mathrm{Ext}^{s,t}_{A}(B_2, \mathbb{F}_2)$$

where E^{May}_{1,B_2} is given by

$$\frac{\mathbb{F}_2[h_{i,j} \mid i > 0, j \geq 0]}{(h_{1,0}, h_{1,1}, h_{1,2}, h_{2,0}, h_{2,1})}$$

There are no non-trivial *May* differentials on the element represented by $h_{3,0}$, since $d_1(h_{3,0}) = 0$ and there are no non-zero targets in the range for higher differentials. The element $h_{3,0}$ survives the *May* spectral seqeunce. There are also no non-trivial differentials on $h_{3,0}$ in the *aAHSS*. Hence $h_{3,0}$ represents a non-zero element in $Ext^{6,1}_{A}(B_2, B_2)$.

Using change of rings we can write

$$\mathrm{Ext}_{A(2)}(B(2), B(2)) = \mathrm{Ext}_{E(Q_2)}(\mathbb{F}_2, B(2)) = \bigoplus_{\mathcal{G}} \mathrm{Ext}_{E(Q_2)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[\tilde{v}_2] \otimes \bigoplus_{c \in \mathcal{G}} \Sigma^{|c|}\mathbb{F}_2 \tag{2}$$

where the second equality follows from $B(2)$ being a trivial $E(Q_2)$-module. We also calculate the same group using the following two spectral sequences

$$\bigoplus_{\mathcal{G}} \mathrm{Ext}^{s,f}_{A(2)}(B(2), \mathbb{F}_2) \implies \mathrm{Ext}^{s,f}_{A(2)}(B(2), B(2))$$
$$E^{\mathrm{May}}_{1,B(2)} \coloneqq \mathrm{Ext}^{s,f}_{\mathrm{gr}(A(2))}(\mathrm{gr}(B(2)), \mathbb{F}_2) \implies \mathrm{Ext}^{s,f}_{A(2)}(B(2), \mathbb{F}_2)$$

wher $E^{\mathrm{May}}_{1,B(2)}$ is given by

$$\mathbb{F}_2[h_{3,0}]$$

We have a commutative diagram of spectral sequences

$$\begin{tikzcd}
E^{\mathrm{May}}_{1,B_2} \arrow[r, Rightarrow] \arrow[d, "\pi"'] & \mathrm{Ext}^{s,f}_{A_2}(B_2, \mathbb{F}_2) \arrow[d, "\mathrm{res}"] \\
E^{\mathrm{May}}_{1,B(2)} \arrow[r, Rightarrow] & \mathrm{Ext}^{s,f}_{A(2)}(B(2), \mathbb{F}_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
\oplus_{\mathcal{G}} \mathrm{Ext}^{s,f}_{A}(B_2, \mathbb{F}_2) \arrow[r, Rightarrow] \arrow[d, "\mathrm{res}"'] & \mathrm{Ext}^{s,f}_{A}(B_2, B_2) \arrow[d, "k"] \\
\oplus_{\mathcal{G}} \mathrm{Ext}^{s,f}_{A(2)}(B(2), \mathbb{F}_2) \arrow[r, Rightarrow] & \mathrm{Ext}^{s,f}_{A(2)}(B(2), B(2))
\end{tikzcd}$$

Under the map π, the element $h_{3,0}$ goes to itself. It also represents the element $\overline{v}_2$ in $\mathrm{Ext}^{1,7}_{A}(B_2, B_2)$. From 2, we know that $\tilde{v}_2$ is the only element in $\mathrm{Ext}^{1,7}_{A(2)}(B(2), B(2))$. This shows that $\overline{v}_2$ maps to $\tilde{v}_2$ under the map k.

$\square$

We extend this result to the $\mathbb{R}$-motivic case.

Theorem 5.2. Let $\widetilde{B}_2^{\mathbb{R}}$ denote any $\mathcal{A}^{\mathbb{R}}$-module which restricts as an $\mathcal{A}^{\mathbb{R}}(2)$-module to $\widetilde{B}_2^{\mathbb{R}}$. Then there exists an element $\overline{v}_2 \in \mathrm{Ext}^{6,1}_{\mathcal{A}^{\mathbb{R}}}(\widetilde{B}_2^{\mathbb{R}}, \widetilde{B}_2^{\mathbb{R}})$ which maps to $\tilde{v}_2$ under the map

$$k : \mathrm{Ext}^{s,t}_{\mathcal{A}^{\mathbb{R}}}(\widetilde{B}_2^{\mathbb{R}}, \widetilde{B}_2^{\mathbb{R}}) \to \mathrm{Ext}^{s,t}_{\mathcal{A}^{\mathbb{R}}(2)}(\widetilde{\mathcal{B}}^{\mathbb{R}}(2), \widetilde{\mathcal{B}}^{\mathbb{R}}(2))$$

Proof. We use *May*'s spectral sequence to write

$$E_{1,B_2^{\mathbb{C}}}^{\mathrm{May}} \coloneqq \mathrm{Ext}^{s,f,w}_{\mathrm{gr}(\mathcal{A}^{\mathbb{C}}}(\mathrm{gr}(B_2^{\mathbb{C}}), \mathbb{M}_2^{\mathbb{C}}) \implies \mathrm{Ext}(B_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$$

where $E_{1,B_2^{\mathbb{C}}}^{\mathrm{May}}$ is given by

$$\frac{\mathbb{M}_2^{\mathbb{C}}[h_{i,j} \mid i > 0, j \geq 0]}{(h_{1,0}, h_{1,1}, h_{1,2}, h_{2,0}, h_{2,1})}$$

The element $h_{3,0}$ survives the *May* spectral sequence and represents an element in $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(B_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$, which we continue to call $h_{3,0}$. The *ρ-Bockstein* spectral sequnces allows us to approximate the $\mathbb{R}$-motivic Ext groups.

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{C}}}(B_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho] \implies \mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}}(\widetilde{B}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$$

There is no ρ-multiple of an element in $(5,2)$ for a possible target of a differential on $h_{3,0}$ which has motivic weight 1. $h_{3,0}$ has motivic weight 3 and a non-zero differential on it should land on an element in $(5,2)$ with the same motivic weight. This is not a possibility ρ-multiples of elements in filtration 2 that land in stem 5 will have motivic weight less than 3. So $h_{3,0}$ survives the ρ-Bockstein spectral sequence. We will contitute to use $h_{3,0}$ to denote the element it represents in $\mathrm{Ext}^{6,1,3}_{\mathcal{A}^{\mathbb{R}}}(\widetilde{B}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$.

We make use of *aAHSS* to find,

$$\bigoplus_{\mathcal{G}} \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\widetilde{B}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}) \implies \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\widetilde{B}_2^{\mathbb{R}}, \widetilde{B}_2^{\mathbb{R}})$$

There are no elements in $\mathrm{Ext}^{5,2,*}_{\mathcal{A}^{\mathbb{C}}}(B_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$ of motivic weight greater than or equal to 3. Hence there are no possible *aAHSS* differentials on $h_{3,0} \otimes \mathrm{gr}_i^{\mathrm{cell}}(\widetilde{B}_2^{\mathbb{R}}) \in \bigoplus_{\mathcal{G}} \mathrm{Ext}^{6,1,*}_{\mathcal{A}^{\mathbb{R}}}(\widetilde{B}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$. This tells us that $h_{3,0}$ survives the *aAHSS* and represents an element $\overline{v}_2 \in \mathrm{Ext}^{6,1,3}_{\mathcal{A}^{\mathbb{R}}}(\widetilde{B}_2^{\mathbb{R}}, \widetilde{B}_2^{\mathbb{R}})$.

Similar to the previous proof, using change of rings we can write

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(2)}(\widetilde{B}_2^{\mathbb{R}}, \widetilde{B}_2^{\mathbb{R}}) = \mathrm{Ext}_{E(\tilde{Q}_2)}(\mathbb{M}_2^{\mathbb{R}}, B(2)) = \bigoplus_{\mathcal{G}} \mathrm{Ext}_{E(\tilde{Q}_2)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}) = \mathbb{M}_2^{\mathbb{R}}[\tilde{v}_2] \otimes \bigoplus_{c \in \mathcal{G}} \Sigma^{|c|}\mathbb{M}_2^{\mathbb{R}} \tag{3}$$

where the second equality follows from $\widetilde{B}_2^{\mathbb{R}}$ being a trivial $E(\tilde{Q}_2)$-module. We also calculate the same group using the same spectral sequences as before: *May, ρ-Bockstein, aAH*. The E_1 page for the *May* spectral sequence is given by

$$\mathbb{M}_2^{\mathbb{C}}[h_{3,0}]$$

One can see that the element $h_{3,0}$ survives the spectral sequences and is represented by the element $\tilde{v}_2$ in $\mathrm{Ext}^{6,1,3}_{\mathcal{A}^{\mathbb{R}}(2)}(\widetilde{B}_2^{\mathbb{R}}, \widetilde{B}_2^{\mathbb{R}})$. The same techniques outlined towards the end of proof of 5.1 using commutativity of spectral sequences extend to the $\mathbb{R}$-motivic case. □

Proof. We will give an alternate proof of the same statement using free resolutions of the modules $\widetilde{B}_2^{\mathbb{R}}$ and $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$.

$$\cdots \to \Sigma^{14}\mathcal{A}^{\mathbb{R}}(2) \to \Sigma^{7}\mathcal{A}^{\mathbb{R}}(2) \to \mathcal{A}^{\mathbb{R}}(2) \to \widetilde{\mathcal{B}}^{\mathbb{R}}(2)$$

is a minimal free $\mathcal{A}^{\mathbb{R}}(2)$-module resolution of $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$. Dualizing the resolution in order to find $\mathrm{Ext}^{1,7}$, we have

$$\hom_{\mathcal{A}^{\mathbb{R}}(2)}(\mathcal{A}^{\mathbb{R}}(2), \Sigma^7\widetilde{\mathcal{B}}^{\mathbb{R}}(2)) \to \hom_{\mathcal{A}^{\mathbb{R}}(2)}(\Sigma^7\mathcal{A}^{\mathbb{R}}(2), \Sigma^7\widetilde{\mathcal{B}}^{\mathbb{R}}(2)) \to \cdots \tag{4}$$

An $\mathcal{A}^{\mathbb{R}}$-module resolution of $\widetilde{B}_2^{\mathbb{R}}$ looks like

$$\cdots \to \Sigma^{14}\mathcal{A}^{\mathbb{R}} \oplus P_2 \to \Sigma^7\mathcal{A}^{\mathbb{R}} \oplus P_1 \to \mathcal{A}^{\mathbb{R}} \to \widetilde{B}_2^{\mathbb{R}}$$

where P_i are free $\mathcal{A}^{\mathbb{R}}$-modules in degree 9 or higher. Dualizing to find $\mathrm{Ext}^{1,7}$, we have

$$\hom_{\mathcal{A}^{\mathbb{R}}}(\mathcal{A}^{\mathbb{R}}, \Sigma^7\widetilde{B}_2^{\mathbb{R}}) \to \hom_{\mathcal{A}^{\mathbb{R}}}(\Sigma^7\mathcal{A}^{\mathbb{R}} \oplus P_1, \Sigma^7\widetilde{B}_2^{\mathbb{R}}) \to \cdots \tag{5}$$

$\mathcal{A}^{\mathbb{R}}$ is a free $\mathcal{A}^{\mathbb{R}}(2)$-module and $\widetilde{B}_2^{\mathbb{R}}$ restricted to $\mathcal{A}^{\mathbb{R}}(2)$ is the module $\widetilde{\mathcal{B}}^{\mathbb{R}}(2)$. The restriction map k is then induced by restriction map from the resolution 4 to resolution 5. This map between hom groups is surjective at each level, so the map

$$k : \mathrm{Ext}^{1,7}_{\mathcal{A}^{\mathbb{R}}}(\widetilde{B}_2^{\mathbb{R}}, \widetilde{B}_2^{\mathbb{R}}) \to \mathrm{Ext}^{1,7}_{\mathcal{A}^{\mathbb{R}}(2)}(\widetilde{\mathcal{B}}^{\mathbb{R}}(2), \widetilde{\mathcal{B}}^{\mathbb{R}}(2))$$

is also surjective, meaning that we can find a $\overline{v}_2$ that maps to $\tilde{v}_2$ under the map k. $\square$

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